Compositional Structural Metrics on Atomic Transition Forms

1. Introduction

1.1 From Structural Algebra to Quantitative Structure

Finite deterministic computations admit multiple structural representations.
Straight-line programs (SLPs) provide a classical compressed representation of strings and computations and have been extensively studied in algorithmic contexts [5].
Among the most common are dependency graphs, arithmetic circuits, and straight-line programs.
While these representations differ syntactically, they share a common structural core: computations are composed of elementary operations arranged according to a directed acyclic dependency structure.
In previous work, we introduced **atomic forms** as canonical structural representations of deterministic transitions over a fixed functional basis.
Atomic forms isolate the elementary operational structure of computations by decomposing composite transitions into bounded atomic dependency fragments.
When considered modulo interface-preserving isomorphism, atomic forms form a well-defined structural space that supports natural operations such as sequential composition and structural embedding.
More precisely, the space of atomic forms over a fixed signature Σ carries a compositional algebraic structure.
Sequential composition induces a monoid structure, while structural embeddings define a natural preorder that captures containment of dependency structures.
Together, these constructions provide a purely structural description of deterministic computations independent of syntactic representation.
However, this algebraic viewpoint remains **qualitative**.
It describes how atomic forms compose and relate structurally, but it does not provide quantitative measures of structural complexity.
In practice and theory alike, computations are often compared using measures such as program length, circuit depth, or parallelism.
Such quantities capture important aspects of computational structure, yet classical graph-based metrics are generally not compatible with compositional structure.
The goal of the present work is therefore to introduce a class of **quantitative invariants** that remain compatible with the structural algebra of atomic forms.
We formalize a class of functions called **compositional structural metrics (CSM)** that measure structural complexity while respecting the fundamental invariances of the atomic representation.
These metrics are designed to satisfy four basic principles:

- invariance under structural isomorphism,
- monotonicity under structural embedding,
- subadditivity under sequential composition,
- independence from representational artifacts.

Together, these conditions ensure that the resulting quantities behave consistently with the algebraic structure of atomic forms.
Dependence on preceding results. The present paper takes as given the atomic normalization framework developed in **[1]**, which produces a canonical atomic representation for SLP-definable deterministic transitions over a fixed signature. We further rely on the algebraic and order-theoretic structure of atomic forms established in **[2]**: the factor space $\boldsymbol{A}_{\Sigma}$ carries a well-

defined sequential composition $\circ$ (associative with identity) and a structural embedding preorder $\preceq$ that descends to equivalence classes.
Our contribution begins at this point: we introduce quantitative invariants $M: \boldsymbol{A}_{\Sigma} \to \mathbb{R}_{\geq 0}$ that are compatible with $\circ$ and $\preceq$, and we study the induced geometric (pseudometric) structure.

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1.2 Structural vs Classical Graph Metrics
Graph theory and algorithmic complexity offer a large variety of structural measures for graphs and computational representations.
Examples include edit distances, spectral invariants, Weisfeiler–Lehman refinement measures, and various combinatorial graph parameters.
While such metrics are useful for many purposes, they are generally **not compatible with compositional computational structure**.
For instance, graph edit distance measures the minimal sequence of local graph modifications required to transform one graph into another.
However, edit operations typically do not respect interface structure and are not stable under sequential composition of computations.
Spectral graph invariants depend on eigenvalues of adjacency or Laplacian matrices.
These invariants capture global structural properties but are insensitive to ordered interfaces and compositional semantics.
Similarly, Weisfeiler–Lehman refinement and related graph isomorphism procedures are designed for graph distinguishability rather than structural measurement compatible with composition.
In contrast, the invariants introduced in this work are **specifically designed for compositional computational structures**.
They are defined on atomic forms modulo interface-preserving isomorphism and interact coherently with both sequential composition and structural embedding.
Thus the present framework should not be viewed as an alternative to classical graph metrics, but rather as a **specialized quantitative layer for compositional dependency structures**.

1.3 Compositional Structural Metrics
We introduce the notion of a **compositional structural metric (CSM)** on the space of atomic forms.
Informally, a compositional structural metric is a function

$$M: \boldsymbol{A}_{\Sigma} \to \mathbb{R}_{\geq 0}$$

that measures structural complexity while remaining compatible with the compositional algebra of atomic forms.
Formally, a function M is a compositional structural metric if it satisfies:

1. **Isomorphism invariance**

Structural isomorphisms preserve the value of the metric.

2. **Embedding monotonicity**

If one atomic form structurally embeds into another, the metric cannot decrease.

3. **Subadditivity under composition**
The complexity of a sequential composition does not exceed the sum of complexities of its components.
4. **Representation independence**
The metric depends only on the canonical atomic representation and not on syntactic realization.
These axioms capture the essential compatibility requirements between quantitative measurement and compositional structure.

1.4 Fundamental Structural Metrics
Two basic examples of compositional structural metrics arise naturally from the underlying dependency structure.
The first invariant is **atomic length**

$$L(A) = |V(A)|$$

which counts the number of atomic computational vertices.
The second invariant is **atomic depth**

$$D(A) = \text{maxpath length in A}$$

which measures the length of the longest dependency chain.
Both quantities are classical structural parameters for directed acyclic graphs.
However, when defined on atomic forms modulo interface-preserving isomorphism, they satisfy all axioms of compositional structural metrics.
In particular, both invariants are monotone under structural embedding and subadditive with respect to sequential composition.
These two quantities represent complementary aspects of computational structure.
Length captures the **total amount of atomic work**, while depth measures the **critical dependency path**.

1.5 Derived Structural Quantities
Using the fundamental invariants L and D, one can construct derived structural quantities.
One important example is **structural parallelism**

$$D(A) = \frac{L(A)}{D(A)}.$$

This quantity informally measures the amount of work available per dependency layer.
However, structural parallelism does not generally satisfy the axioms of compositional structural metrics.
In particular, it need not be monotone under structural embedding and does not satisfy subadditivity under composition.
Thus P is best viewed as a **derived invariant constructed from compositional metrics**, rather than as a compositional structural metric itself.
This distinction highlights that the class of CSMs forms a restricted family of invariants specifically adapted to compositional structure.

1.6 Trade-offs Between Structural Metrics

Different compositional structural metrics capture different aspects of structural complexity.
In particular, it is generally impossible to minimize both atomic length and atomic depth simultaneously.
We establish the existence of atomic forms exhibiting **trade-offs between length and depth**, showing that the two invariants represent fundamentally distinct optimization objectives.
More generally, we show that no single compositional structural metric from a broad class of additive and embedding-monotone invariants induces a total ordering on the space of atomic forms.
These results illustrate the intrinsic multi-dimensional nature of structural complexity.

1.7 Closure Properties of Compositional Structural Metrics

The class of compositional structural metrics possesses a natural closure property.
If M_1 and M_2 are compositional structural metrics and $\alpha, \beta \geq 0$, then the linear combination

$$\alpha M_1 + \beta M_2$$

is again a compositional structural metric.
Consequently, the space of compositional structural metrics forms a **convex cone** in the vector space of real-valued functions on atomic forms.
This structure allows one to construct families of structural invariants tailored to specific applications while preserving compatibility with composition and embedding.

1.8 Induced Pseudometric Structures

Every compositional structural metric naturally induces a pseudometric on the space of atomic forms.
Given a $CSM\ M$, define

$$d_M(A_1, A_2) = |M(A_1) - M(A_2)|.$$

We show that d_m satisfies the axioms of a pseudometric.
When atomic forms are considered modulo equality of the invariant M, this pseudometric becomes a genuine metric on the resulting factor space.
These constructions equip the structural space of atomic forms with natural quantitative geometries induced by compositional invariants.

1.9 Relation to Parallel Evaluation

The invariants introduced here admit a natural interpretation in the context of parallel computation.
Atomic length corresponds to the total amount of work performed by a computation, while atomic depth represents the critical dependency path that limits parallel execution.
This interpretation parallels classical results such as Brent's theorem relating work and parallel time in parallel algorithms.
However, the present work focuses exclusively on **structural properties of dependency representations**, without making claims about computational complexity or execution models.

1.10 Contributions

The main contributions of this work can be summarized as follows:

- introduction of the class of compositional structural metrics on atomic forms,
- proof that atomic length and atomic depth are compositional structural metrics,
- identification of derived invariants such as structural parallelism,
- trade-off results showing incompatibility between certain structural objectives,
- closure of the class of CSMs under nonnegative linear combinations,
- construction of pseudometric structures induced by compositional metrics.

Together, these results provide a quantitative framework for studying the structural complexity of atomic dependency representations.

2. Preliminaries

The structural interpretation of L (work) and D (critical path) parallels classical viewpoints in parallel evaluation and dependency DAG scheduling (e.g., **[3,4]**), although we remain purely structural.

2.1 Imported structural framework

Imported structural framework. We work on the quotient space $\boldsymbol{A}_{\Sigma}$ of atomic forms modulo interface-preserving isomorphism. The construction of atomic forms as canonical representatives of SLP-definable deterministic transitions is developed in **[1]**. The present paper does not revisit normalization; instead, we treat atomic forms as our primitive structural objects.

The algebraic operation of sequential composition $\circ$ on ($\boldsymbol{A}_{\Sigma}$ and its well-definedness on equivalence classes are established in **[2]**. Likewise, the structural embedding relation $\preceq$\preceq$\preceq$ and its preorder properties (reflexivity and transitivity, with antisymmetry modulo isomorphism) are established in **[2]**.

In what follows we use only these consequences: $(\boldsymbol{A}_{\Sigma},\circ)$ is a monoid, and (ii) $\preceq$ is a preorder compatible with interface-preserving structure.

2.2 Atomic Forms over a Fixed Signature

Throughout the paper we fix a finite functional signature Σ.

All computational descriptions considered in this work are assumed to use only operations from this fixed basis.

Atomic forms provide canonical structural representations of deterministic computational transitions over the signature Σ.

Intuitively, an atomic form is a finite directed acyclic dependency structure in which each computational vertex represents a single atomic operation whose inputs are connected through ordered ports.

Formally, an **atomic form** over Σ is defined as a finite directed acyclic graph equipped with the following structure:

- a finite set of computational vertices,
- a labeling function assigning to each vertex an operation symbol from the signature,
- an ordered set of input ports for each vertex determined by the arity of the assigned symbol,
- a distinguished ordered set of input interface ports,
- a distinguished ordered set of output interface ports.

Edges connect either interface inputs or vertex outputs to vertex input ports, or connect vertex outputs to interface outputs.
Interface ports themselves are never directly connected to one another.
This ensures that all dependencies terminate either at vertex input ports or at designated output interfaces.
In addition, atomic forms satisfy two structural conditions:

- **port uniqueness**, ensuring that each vertex input port receives exactly one incoming edge, and
- **acyclicity**, ensuring that the directed dependency structure contains no cycles when restricted to vertices.

These conditions guarantee that atomic forms represent well-defined finite dependency structures corresponding to deterministic computation steps.
Atomic forms are considered **up to interface-preserving isomorphism**.
Two atomic forms are said to be isomorphic if there exists a bijection between their vertex sets that preserves:

- directed edges,
- operation labels,
- ordering of input ports at each vertex,
- ordering of input and output interfaces.

Isomorphic atomic forms represent the same structural computation independently of syntactic naming of vertices.
We denote by

$$\mathcal{A}_\Sigma$$

the class of all atomic forms over the signature Σ.

2.3 Factor Space of Atomic Forms

Since atomic forms are considered modulo interface-preserving isomorphism, it is natural to work on the corresponding factor space.
Let

$$\boldsymbol{A}_\Sigma = \mathcal{A}_\Sigma / \cong$$

denote the set of equivalence classes of atomic forms under interface-preserving isomorphism.
Elements of $\boldsymbol{A}_\Sigma$ therefore represent **canonical structural forms** rather than particular syntactic realizations.
All constructions in the present work are defined on this factor space.
Consequently, structural measurements introduced later are automatically independent of vertex naming or other representational artifacts.
Working modulo isomorphism also ensures that structural invariants reflect genuine properties of computational dependency structure.

2.4 Sequential Composition

Atomic forms support a natural notion of sequential composition.
Let

$$A_1, A_2 \in \boldsymbol{A}_\Sigma$$

be atomic forms such that the output interface of A_1 matches the input interface of A_2.
The **sequential composition**

$$A_2 \circ A_1$$

is obtained by connecting the output interface ports of A_1 to the corresponding input interface ports of A_2.

After the interface connection is established, the intermediate interface ports are removed and the resulting dependency structure is interpreted as a new atomic form.

This operation preserves the defining structural properties of atomic forms.

In particular, the resulting graph remains finite and acyclic, and port uniqueness is preserved.

Sequential composition therefore induces a binary operation on the factor space $\boldsymbol{A}_\Sigma$.

As established in previous work, the following properties hold:

- **closure**: the composition of two atomic forms is again an atomic form,
- **associativity**: sequential composition is associative,
- **identity**: the empty wiring between matching interfaces acts as a neutral element.

Consequently, the pair

$$(\boldsymbol{A}_\Sigma, \circ)$$

forms a **monoid** under sequential composition.

This monoidal structure provides the fundamental algebraic framework for the structural space of atomic forms.

2.5 Structural Embedding

In addition to composition, atomic forms admit a natural structural ordering.

Given two atomic forms

$$A_1, A_2 \in \boldsymbol{A}_\Sigma,$$

we say that A_1 **structurally embeds** into A_2, written

$$A_1 \preceq A_2,$$

if there exists an injective mapping between their vertex sets that preserves:

- directed edges,
- operation labels,
- ordering of vertex input ports,
- ordering of interface ports.

Intuitively, this relation means that the dependency structure of A_1 appears as a substructure inside A_2.

Structural embedding respects the canonical representation of atomic forms and is defined independently of syntactic naming of vertices.

The embedding relation satisfies:

- **reflexivity**: every atomic form embeds into itself,
- **transitivity**: embeddings compose.

Therefore $\preceq$ defines a **preorder** on $\boldsymbol{A}_\Sigma$.

When considered modulo isomorphism classes, the preorder induces a natural **partial order on structural forms**, describing containment of dependency structures.

2.6 Structural Reductions

Atomic forms often admit local structural simplifications that preserve computational meaning.

Typical examples include:

- elimination of vertices that do not contribute to any output interface,
- contraction of chains of intermediate vertices representing trivial structural transformations,
- identification of duplicated substructures representing common computational fragments.

Such operations are referred to as **structural reductions**.
Although the present work does not rely on a specific reduction system, we assume that reductions satisfy two basic properties:

1. reductions preserve the semantic behavior represented by the atomic form, and
2. reductions do not increase structural size or depth.

Consequently, structural reductions are **monotone with respect to the embedding preorder**.
In particular, if a reduction transforms A into B, then

$$B \preceq A.$$

This monotonicity property will play an important role in the study of structural metrics.

2.7 Structural Measurements

The purpose of the present work is to introduce quantitative measurements on the structural space $\boldsymbol{A}_\Sigma$.
Such measurements should satisfy several natural requirements.
First, they must be **representation-independent**, meaning that isomorphic atomic forms receive identical values.
Second, they should be **compatible with structural embedding**, ensuring that larger dependency structures are not assigned smaller complexity values.
Third, they should interact coherently with sequential composition.
These principles motivate the definition of **compositional structural metrics**, introduced in the next section.
Before formalizing this concept, we summarize the structural framework established above.
The space $\boldsymbol{A}_\Sigma$ carries:

- a **monoid structure** induced by sequential composition, and
- a **preorder structure** induced by structural embedding.

The goal of the following sections is to introduce quantitative invariants that remain compatible with both of these structural features.
Such invariants will provide a systematic way to measure the structural complexity of atomic dependency representations.

3. Compositional Structural Metrics

The definition below is formulated over the monoid $(\boldsymbol{A}_\Sigma, \circ)$ and the embedding preorder $\preceq$ recalled above. These structures constitute the minimal interface we require from the preceding papers **[1,2]**.

3.1 Definition

The structural space $\boldsymbol{A}_\Sigma$ introduced in the previous section carries both an algebraic structure given by sequential composition and an order structure induced by structural embedding.
To study the structural complexity of atomic forms within this framework, we introduce a class of quantitative invariants compatible with these structures.

Let

$$\boldsymbol{A}_{\Sigma}$$

denote the factor space of atomic forms over the fixed signature Σ\SigmaΣ, considered modulo interface-preserving isomorphism.
A function

$$M: \boldsymbol{A}_{\Sigma} \to \mathbb{R}_{\geq 0}$$

is called a **compositional structural metric (CSM)** if it satisfies the following conditions.

(1) Isomorphism invariance

For any atomic forms $A_1, A_2 \in \boldsymbol{A}_{\Sigma}$,

$$A1 \cong A2 \Rightarrow M(A_1) = M(A_2).$$

This condition ensures that the metric depends only on the structural form of the computation and not on the syntactic naming of vertices or ports.

(2) Embedding monotonicity

For any atomic forms $A_1, A_2 \in \boldsymbol{A}_{\Sigma}$,

$$A_1 \preceq A_2 \;\Rightarrow\; M(A_1) \leq M(A_2).$$

Thus structural containment cannot decrease the value of the metric.
This property reflects the intuition that a larger or more complex dependency structure should not have smaller structural complexity.

(3) Subadditivity under composition

For any composable atomic forms $A_1, A_2 \in \boldsymbol{A}_{\Sigma}$,

$$M(A_2 \circ A_1) \leq M(A_1) + M(A_2).$$

This condition expresses compatibility with sequential composition.
The structural complexity of a composed computation is bounded by the sum of the complexities of its components.
Subadditivity ensures that the metric behaves consistently with the monoidal structure of atomic forms.

(4) Representation independence

The metric M is defined on equivalence classes in $\boldsymbol{A}_{\Sigma}$ and therefore depends only on canonical atomic representations rather than on particular syntactic realizations.
This requirement guarantees that measurements reflect intrinsic structural properties of computations.
A function satisfying the above conditions will be referred to as a **compositional structural metric**.
We denote by $\mathcal{M}_{\Sigma}$

the class of all compositional structural metrics on $\boldsymbol{A}_{\Sigma}$.

3.2 Basic Structural Consequences

The axioms of compositional structural metrics imply several immediate structural properties.

Monotonicity under structural reductions

Suppose a structural reduction transforms an atomic form A into B.
As discussed in Section 2, such reductions satisfy

$$B \preceq A.$$

By embedding monotonicity we obtain

$$M(B) \leq M(A).$$

Thus every compositional structural metric is **monotone with respect to structural simplifications**.
In particular, structural reductions cannot increase the value of a CSM.
This property ensures that structural metrics behave consistently with canonical simplification procedures.

Compatibility with the monoid structure

Let

$$A_1, A_2, A_3 \in \boldsymbol{A}_\Sigma$$

be atomic forms such that the compositions involved are well defined.
Using subadditivity and associativity of composition we obtain

$$M((A_3 \circ A_2) \circ A_1) \leq M(A_1) + M(A_2) + M(A_3).$$

Thus repeated sequential composition produces at most linear growth of the metric with respect to the number of components.
This property ensures that compositional structural metrics provide stable complexity measures for large composed dependency structures.

Stability under isomorphic transformations

Since CSMs are defined on equivalence classes modulo interface-preserving isomorphism, they are invariant under all structural renamings that preserve the computational dependency structure.
Consequently, the value of a compositional structural metric is determined entirely by the canonical structural representation of the computation.

3.3 Structural Interpretation

The axioms defining compositional structural metrics can be interpreted as compatibility requirements with the structural space of atomic forms.
Each condition corresponds to a fundamental structural property:

- **isomorphism invariance** guarantees independence from syntactic representation,
- **embedding monotonicity** ensures compatibility with the structural preorder describing containment of dependency structures,
- **subadditivity** ensures compatibility with sequential composition.

Together these requirements ensure that compositional structural metrics behave consistently with both the algebraic and order-theoretic structures of $\boldsymbol{A}_\Sigma$.
In particular, a compositional structural metric can be viewed as a **complexity measure defined directly on the structural space of atomic forms**.

3.4 Examples and Non-Examples

Not every numerical invariant on atomic forms satisfies the axioms of a compositional structural metric.

Many natural graph parameters violate one or more of the defining conditions.

For instance, certain global graph invariants may fail to be monotone under structural embedding.

Similarly, ratios of structural quantities may violate subadditivity under composition.

The following sections will introduce several concrete examples of compositional structural metrics and illustrate the distinction between admissible metrics and derived structural invariants.

In particular, we will show that two classical quantities associated with dependency graphs — **atomic length** and **atomic depth** — satisfy all axioms of compositional structural metrics.

These invariants will serve as fundamental examples of the framework introduced here.

3.5 Structural Role of CSMs

The introduction of compositional structural metrics provides a quantitative layer over the structural algebra of atomic forms.

While the monoid structure describes how computations compose and the embedding preorder captures structural containment, CSMs measure the structural complexity of atomic forms in a way that respects both of these relations.

Thus the triple

$$(\boldsymbol{A}_{\Sigma}, \circ, \preceq)$$

equipped with compositional structural metrics forms a framework in which structural dependency representations can be studied both qualitatively and quantitatively.

The next sections will develop concrete examples of such metrics and analyze their structural properties.

4. Fundamental Examples

The definition of compositional structural metrics introduced in the previous section provides an abstract framework for quantitative invariants on the structural space of atomic forms.

We now present two fundamental examples that arise naturally from the dependency structure of atomic computations.

These examples correspond to two classical measures of computational structure:

- the **total number of computational operations**, and
- the **length of the longest dependency chain**.

When defined on atomic forms modulo interface-preserving isomorphism, these quantities satisfy all axioms of compositional structural metrics.

4.1 Atomic Length

The first structural invariant measures the total number of computational vertices in an atomic form.

Definition 4.1 (Atomic Length)

For an atomic form

$$A \in \boldsymbol{A}_{\Sigma}$$

the **atomic length** of A is defined as

$$L(A) = |V(A)|$$

where $V(A)$ denotes the vertex set of the atomic form.
Thus atomic length counts the total number of atomic computational operations appearing in the dependency structure.

Interpretation

Atomic length represents the **structural size** of a computation.
Since each vertex corresponds to an elementary operation from the signature Σ, the value $L(A)$ measures the total amount of atomic work present in the computation.
This invariant corresponds to the classical notion of **program length** or **circuit size** in computational models.

Theorem 4.1

Atomic length is a compositional structural metric.

Proof

We verify the four defining properties of compositional structural metrics.

(1) Isomorphism invariance

If

$$A_1 \cong A_2$$

then there exists a bijection between the vertex sets of the two atomic forms preserving edges, labels, and port ordering.
In particular,

$$|V(A_1)| = |V(A_2)|.$$

Therefore

$$L(A_1) = L(A_2).$$

(2) Embedding monotonicity

Suppose

$$A_1 \preceq A_2.$$

By definition of structural embedding, there exists an injective mapping

$$\varphi: V(A_1) \hookrightarrow V(A_2).$$

Since the mapping is injective,

$$|V(A_1)| \leq |V(A_2)|.$$

Therefore

$$L(A_1) \leq L(A_2).$$

(3) Subadditivity under composition

Consider two composable atomic forms

$$A_1, A_2 \in \boldsymbol{A}_\Sigma$$

.

The sequential composition $A_1 \circ A_2$ is constructed by connecting interface ports without introducing new computational vertices.
Consequently the vertex set of the composite is the disjoint union

$$V(A_2 \circ A_1) = V(A_1) \sqcup V(A_2).$$

Therefore

$$L(A_2 \circ A_1) = |V(A_1)| + |V(A_2)| = L(A_1) + L(A_2).$$

In particular,

$$L(A_2 \circ A_1) \leq L(A_1) + L(A_2).$$

(4) Representation independence

Since atomic length depends only on the cardinality of the vertex set, and atomic forms are considered modulo isomorphism, the value $L(A)$ depends only on the equivalence class in $\boldsymbol{A}_{\Sigma}$.

All axioms of compositional structural metrics are satisfied.

Therefore atomic length is a compositional structural metric. ■

4.2 Atomic Depth

The second fundamental invariant measures the length of the longest dependency chain in an atomic form.

Definition 4.2 (Atomic Depth)

Let

$$A \in \boldsymbol{A}_{\Sigma}$$

be an atomic form.

The **atomic depth** of A is defined as the maximum length of a directed path in the vertex dependency graph:

$$D(A) = max\{k \mid there\ exists\ a\ directed\ path\ of\ k\ vertices\ in\ A\}.$$

Equivalently, atomic depth is the length of the longest sequence of vertices

$$v_1 \to v_2 \to \cdots \to v_k$$

such that each vertex depends directly on the previous one.

Interpretation

Atomic depth measures the **maximum dependency chain** in the computation.

This quantity corresponds to the classical notion of **critical path length** in dependency graphs or **circuit depth** in circuit complexity.

Depth determines the maximal degree of sequential dependency present in the computation and therefore represents the minimal number of sequential evaluation stages required by the dependency structure.

Theorem 4.2

Atomic depth is a compositional structural metric.

Proof

We verify the defining properties of compositional structural metrics.

(1) Isomorphism invariance

If

$$A_1 \cong A_2$$

then the two atomic forms have identical dependency structures up to renaming of vertices.

Directed paths correspond under the isomorphism, and therefore the maximal path length is preserved.
Hence
$$D(A_1) = D(A_2).$$
(2) Embedding monotonicity
Suppose
$$A_1 \preceq A_2.$$
Then there exists an injective structure-preserving mapping from A_1 into A_2.
Every directed path in A_1 is mapped to a directed path in A_2.
Therefore the maximal path length cannot decrease.
Thus
$$D(A_1) \leq D(A_2).$$
(3) Subadditivity under composition
Consider composable atomic forms A_1 and A_2.
In the sequential composition
$$A_2 \circ A_1$$
every dependency path belongs to one of the following types:

- a path entirely contained in A_1,
- a path entirely contained in A_2,
- a path passing through the interface connection from A_1 into A_2.

Paths of the third type consist of a path in A_1 followed by a path in A_2.
Hence their length is at most
$$D(A_1) + D(A_2).$$
Consequently,
$$D(A_2 \circ A_1) \leq D(A_1) + D(A_2).$$
(4) Representation independence
Atomic depth depends only on the directed dependency structure of the atomic form and is therefore invariant under interface-preserving isomorphism.
Thus atomic depth satisfies all axioms of compositional structural metrics. ■

4.3 Structural Parallelism (Derived Invariant)

Using the two fundamental metrics L and D, we may define derived structural quantities.
One particularly natural invariant is **structural parallelism**, defined as
$$P(A) = \frac{L(A)}{D(A)}.$$
This quantity measures the average amount of computational work per dependency layer.
Informally, it captures the degree of potential parallelism present in the dependency structure.
However, structural parallelism is **not generally a compositional structural metric**.
In particular:

- it need not be monotone under structural embedding, and
- it does not satisfy subadditivity under composition.

Thus P should be viewed as a **derived structural invariant constructed from compositional structural metrics**, rather than as a compositional structural metric itself.
This distinction illustrates that the class of compositional structural metrics forms a restricted family of invariants specifically adapted to the compositional structure of atomic forms.

4.4 Summary

The invariants introduced above provide the fundamental examples of compositional structural metrics.
In particular:

- **atomic length** measures the total number of atomic operations,
- **atomic depth** measures the longest dependency chain.

Both quantities satisfy the axioms of compositional structural metrics and therefore interact coherently with the compositional algebra of atomic forms.
In the following sections we will extend this framework by introducing additional examples of compositional structural metrics and studying the structural relations between different invariants.

5. Extended Examples of Compositional Structural Metrics

The previous section introduced two fundamental compositional structural metrics: atomic length and atomic depth.
These invariants arise directly from basic properties of dependency structures and serve as canonical measures of structural size and sequential dependency.
However, the class of compositional structural metrics is considerably richer.
In this section we present several additional examples illustrating how more refined structural measurements can be constructed while preserving compatibility with the compositional framework.
These examples demonstrate that compositional structural metrics may incorporate additional information such as operation weights, structural width, or input-output complexity.

5.1 Weighted Length

In many computational settings different operations have different structural costs.
For example, arithmetic operations may have varying implementation complexity, or certain primitives may represent larger computational units.
To model such situations, we introduce a weighted version of atomic length.
Let

$$w: \Sigma \to \mathbb{R}_{\geq 0}$$

be a nonnegative weight function assigning a structural cost to each operation symbol in the signature.

Definition 5.1 (Weighted Length)

For an atomic form

$$A \in \boldsymbol{A}_{\Sigma}$$

the **weighted length** of A is defined as

$$L_w(A) = \sum_{v \in V(A)} w(l_A(v))$$

Thus each vertex contributes a weight determined by the label of the operation it represents.

Interpretation

Weighted length measures the **total weighted structural cost** of a computation.
While atomic length treats all operations equally, weighted length allows different operations to contribute differently to structural complexity.
This generalization is particularly useful when the signature contains operations of varying intrinsic complexity.

Proposition 5.1

Weighted length is a compositional structural metric.

Proof

We verify the defining properties.

Isomorphism invariance.

Interface-preserving isomorphisms preserve vertex labels and therefore preserve the sum defining L_w.

Embedding monotonicity.

If $A_1 \preceq A_2$, the embedding induces an injective mapping of vertex sets preserving labels.
Since weights are nonnegative,

$$L_w(A_1) \le L_w(A_2).$$

Subadditivity under composition.

Sequential composition does not introduce new vertices.
Therefore

$$L_w(A_2 \circ A_1) = L_w(A_1) + L_w(A_2).$$

Thus

$$L_w(A_2 \circ A_1) \le L_w(A_1) + L_w(A_2).$$

Representation independence.

The definition depends only on labels and vertex multiplicities, which are invariant under isomorphism.
Hence L_w satisfies the axioms of compositional structural metrics. ■

5.2 Weighted Depth

Depth may also be generalized by assigning weights to operations.
Instead of counting the number of vertices along a dependency chain, we measure the total weight of operations along the chain.

Definition 5.2 (Weighted Depth)

Let

$$w: \Sigma \to \mathbb{R}_{\ge 0}$$

be a weight function.
The **weighted depth** of an atomic form A is defined as

$$D_w(A) = \max_{directed\ \ paths\ P} \sum_{v \in P} w(l_A(v))$$

Thus the weight of a path equals the sum of weights of the vertices along the path.

Interpretation
Weighted depth measures the **maximum accumulated structural cost along a dependency chain**.
This invariant generalizes atomic depth by allowing different operations to contribute different amounts to the sequential dependency cost.

Proposition 5.2
Weighted depth is a compositional structural metric.

Proof
Isomorphism invariance.
Directed paths correspond under isomorphisms and vertex labels are preserved; therefore weighted path lengths remain unchanged.
Embedding monotonicity.
Every path in A_1 is mapped to a path in A_2 under structural embedding.
Thus

$$D_w(A_1) \leq D_w(A_2).$$

Subadditivity under composition.
Any path in $A_2 \circ A_1$ is either contained within a component or is a concatenation of a path in A_1 followed by a path in A_2.
Thus

$$D_w(A_2 \circ A_1) \leq D_w(A_1) + D_w(A_2).$$

Representation independence.
The definition depends only on path structure and vertex labels.
Therefore D_w satisfies all axioms of compositional structural metrics. ■

5.3 Width-Type Invariants
Another structural property of dependency graphs is the maximal size of a layer of mutually independent operations.
To formalize this concept, we consider level decompositions induced by dependency structure.
For an atomic form A, define the **level** of a vertex as the length of the longest dependency path ending at that vertex.
Vertices with equal levels form a layer of operations that can be executed in parallel once all dependencies from previous layers are satisfied.

Definition 5.3 (Structural Width)
The **width** of an atomic form A is defined as

$$W(A) = \max_k |\{v \in V(A): level(v) = k\}|$$

Thus width measures the maximal number of vertices appearing in the same dependency level.

Interpretation
Structural width measures the **maximum amount of parallel work available at a single stage of the dependency structure**.
This quantity reflects a structural notion of parallel capacity inherent in the computation.

Remark
Unlike length and depth, width does **not necessarily satisfy the subadditivity property under composition**.
Sequential composition may align levels in ways that increase the maximal layer size beyond the sum of the individual widths.
Consequently width-type invariants must be treated with care when considered as candidates for compositional structural metrics.
Certain restricted variants of width may satisfy the CSM axioms, but width in its basic form generally does not.
Thus width is best viewed as a **structural parameter rather than a canonical compositional structural metric**.

5.4 Fan-In and Fan-Out Complexity
Additional structural measurements may be derived from local connectivity properties of dependency graphs.
For example, one may consider the maximum number of inputs used by operations or the number of outputs that propagate from a given vertex.
Let

$$fanin(v)$$

denote the number of incoming dependency edges at vertex v, and let

$$fanout(v)$$

denote the number of outgoing edges.
Since the arity of operations in the signature is fixed, fan-in values are bounded by the signature itself.
However fan-out values depend on the way intermediate results are reused within the dependency structure.

Definition 5.4 (Fan-Out Complexity)
The **fan-out complexity** of an atomic form is defined as

$$F(A) = \max_{v \in V(A)} fanout(v)$$

This invariant measures the maximum number of times the output of a single operation is reused by subsequent operations.

Structural Interpretation

Fan-out complexity captures the degree of **structural sharing** present in the computation.
High fan-out indicates that certain intermediate results are reused multiple times, which may influence memory usage or structural replication in computational realizations.
However, fan-out complexity generally fails to satisfy subadditivity under composition and therefore does not typically qualify as a compositional structural metric.
Nevertheless, such parameters provide useful additional perspectives on structural complexity.

5.5 Summary of Extended Metrics

The examples discussed above illustrate the diversity of quantitative invariants that may be associated with atomic forms.
Some of these invariants, such as weighted length and weighted depth, satisfy the axioms of compositional structural metrics and therefore belong to the class of CSMs.
Others, such as width and fan-out complexity, capture important structural features but fail to satisfy the compositional compatibility conditions required of CSMs.
These examples highlight an important conceptual distinction.
Compositional structural metrics form a **restricted family of invariants specifically adapted to the compositional structure of atomic forms**.
Not all graph-theoretic quantities satisfy the required axioms.
In the next section we will examine structural relationships between different compositional structural metrics and establish trade-off results demonstrating that distinct metrics capture fundamentally different aspects of structural complexity.

6. Trade-offs and Restricted Separation

The compositional structural metrics introduced in the previous sections capture different aspects of structural complexity of atomic forms.
Atomic length measures the total number of computational operations, while atomic depth measures the maximal dependency chain within the computation.
These invariants reflect distinct structural properties of dependency graphs.
Consequently, optimizing one metric does not necessarily optimize the other.
In this section we show that these metrics may exhibit **structural trade-offs**, meaning that improvements in one metric may require deterioration in another.
We also establish limited separation results showing that no single compositional structural metric from a broad class can induce a total ordering on the space of atomic forms.
These results illustrate the intrinsically multi-dimensional nature of structural complexity.

6.1 Trade-off Between Length and Depth

The first result demonstrates that atomic length and atomic depth capture fundamentally different structural features.

Intuitively, a computation can be organized either as a long sequential chain with minimal total size or as a wider structure with shorter dependency chains but larger total size.

Theorem 6.1 (Length–Depth Trade-off)
There exist atomic forms A and B such that

$$L(A) < L(B)$$

but

$$D(A) > D(B).$$

Proof
We construct two families of atomic forms exhibiting opposite structural characteristics.
Let A be a linear dependency chain consisting of k vertices arranged sequentially

$$v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k.$$

In this case,

$$L(A) = k$$

and

$$D(A) = k.$$

Thus the computation has minimal structural width but maximal sequential dependency.
Now consider an atomic form B consisting of a balanced dependency structure of m vertices arranged in several parallel layers.
For example, one may construct a binary-tree-like structure where each layer contains multiple vertices that depend only on vertices from the previous layer.
Such a structure can have depth approximately

$$D(B) \approx \log m$$

while containing

$$L(B) = m$$

vertices.
Choosing parameters such that

$$k < m$$

but

$$k > \log m$$

yields

$$L(A) < L(B)$$

and

$$D(A) > D(B).$$

Thus the two atomic forms exhibit opposite ordering with respect to length and depth. ■

Interpretation
This result shows that minimizing atomic length and minimizing atomic depth represent **distinct structural optimization objectives**.

Reducing the total number of operations may increase sequential dependency, while reducing dependency chains may require additional structural replication.
Therefore no single structural invariant can simultaneously minimize both quantities in general.

6.2 Restricted Metric Incomparability
We now consider a broader class of compositional structural metrics.
Suppose we consider metrics constructed from additive structural quantities such as:

- atomic length,
- atomic depth,
- weighted variants of these metrics.

These metrics satisfy the axioms of compositional structural metrics and therefore respect the compositional structure of atomic forms.
However, even within this restricted class, no single invariant can completely order the space of atomic forms.

Proposition 6.1
For the standard additive compositional structural metrics considered in this work — including atomic length, atomic depth, and their weighted variants — the induced ordering on atomic forms is not total.

Proof Sketch
Consider again the two atomic forms A and B constructed in Theorem 6.1.
These forms satisfy

$$L(A) < L(B)$$

and

$$D(A) > D(B).$$

Thus neither form dominates the other simultaneously with respect to both invariants.
Weighted variants of these metrics preserve the same structural ordering because the trade-off arises from the topology of the dependency structure rather than from the specific weights assigned to operations.
Therefore no single invariant from this class induces a total ordering of atomic forms. ■

6.3 Incomparability Under Finite Families of Metrics
The previous result shows that individual metrics do not totally order the space of atomic forms.
A natural question is whether a **finite collection of metrics** could jointly induce a total ordering.
In other words, one might attempt to compare atomic forms using vectors of structural measurements.
The following observation shows that even this approach is limited.

Proposition 6.2
For any fixed finite family of standard compositional structural metrics

$$M1, M2, \dots, M_k,$$

there exist atomic forms A and B such that

$$M_i(A) \le M_i(B)$$

for some metrics and

$$M_j(A) \le M_j(B)$$

for others.
Thus the pair (A, B) remains incomparable with respect to the combined metric family.

Proof Sketch

Consider families of atomic forms with varying structural shapes, such as chains, balanced trees, and layered dependency structures.
These structures exhibit different scaling behavior with respect to length, depth, and weighted variants of these quantities.
By adjusting the size and layering of the constructions, one can ensure that improvements with respect to one metric necessarily degrade another metric within the chosen family.
Since the family of metrics is finite, such constructions can always be arranged so that no form dominates the other simultaneously for all metrics.
Therefore the induced ordering remains partial. ■

6.4 Structural Consequences

The results of this section demonstrate that structural complexity of atomic forms cannot be captured by a single scalar invariant.
Different compositional structural metrics measure fundamentally different aspects of dependency structure.
Consequently, the structural space of atomic forms should be viewed as **multi-dimensional**, with multiple independent complexity axes.
This observation motivates the study of families of compositional structural metrics and their algebraic properties.
In particular, the next section will show that the class of compositional structural metrics is closed under nonnegative linear combinations, forming a convex cone of structural invariants.

7. Closure of the Class of Compositional Structural Metrics

The previous sections introduced compositional structural metrics and presented several examples, including atomic length, atomic depth, and their weighted variants.
We also observed that different metrics capture different aspects of structural complexity.
In this section we establish a fundamental structural property of the class of compositional structural metrics.
Namely, we show that this class is closed under nonnegative linear combinations.
This result implies that compositional structural metrics form a **convex cone** in the vector space of real-valued functions on the space of atomic forms.
Such closure properties are important because they allow the systematic construction of new structural invariants from existing ones.

7.1 Convex Cone Property

Let

$$\mathcal{M}_\Sigma$$

denote the class of all compositional structural metrics on the structural space $\boldsymbol{A}_\Sigma$.
Recall that every metric in $\mathcal{M}_\Sigma$ satisfies the following properties:

- isomorphism invariance,
- embedding monotonicity,
- subadditivity under composition,
- representation independence.

We now show that these properties are preserved under nonnegative linear combinations.

Theorem 7.1 (Convex Cone of Compositional Structural Metrics)
Let M_1 and M_2 be compositional structural metrics on $\boldsymbol{A}_\Sigma$.
Let

$$\alpha, \beta \geq 0.$$

Define

$$M(A) = \alpha M_1(A) + \beta M_2(A).$$

Then M is also a compositional structural metric.

Proof
We verify each defining property.

(1) Isomorphism invariance
Suppose $$A_1 \cong A_2.$$

Since M_1 and M_2 are compositional structural metrics, we have

$$M_1(A_1) = M_1(A_2)$$

and

$$M_2(A_1) = M_2(A_2).$$

Therefore

$$M(A_1) = \alpha M_1(A_1) + \beta M_2(A_1) = \alpha M_1(A_2) + \beta M_2(A_2) = M(A_2).$$

Thus M is invariant under isomorphism.

(2) Embedding monotonicity
Suppose $$A_1 \preceq A_2.$$

Since M_1 and M_2 satisfy embedding monotonicity, we have

$$M_1(A_1) \leq M_1(A_2)$$

and

$$M_2(A_1) \leq M_2(A_2).$$

Multiplying by nonnegative coefficients and summing gives

$$\alpha M_1(A_1) + \beta M_2(A_1) \leq \alpha M_1(A_2) + \beta M_2(A_2).$$

Therefore

$$M(A_1) \le M(A_2).$$

Thus M satisfies embedding monotonicity.

(3) Subadditivity under composition

Let A_1 and A_2 be composable atomic forms.
Since M_1 and M_2 satisfy subadditivity, we have

$$M_1(A_2 \circ A_1) \le M_1(A_1) + M_1(A_2)$$

and

$$M_2(A_2 \circ A_1) \le M_2(A_1) + M_2(A_2).$$

Multiplying by the coefficients and summing yields

$$M(A_2 \circ A_1) = \alpha M_1(A_2 \circ A_1) + \beta M_2(A_2 \circ A_1)$$

$$\le \alpha(M_1(A_1) + M_1(A_2)) + \beta(M_2(A_1) + M_2(A_2)).$$

Rearranging terms gives

$$M(A_2 \circ A_1) \le (\alpha M_1(A_1) + \beta M_2(A_1)) + (\alpha M_1(A_2) + \beta M_2(A_2)).$$

Thus

$$M(A_2 \circ A_1) \le M(A_1) + M(A_2).$$

Therefore M satisfies subadditivity.

(4) Representation independence

Since both M_1 and M_2 are defined on equivalence classes of atomic forms, the linear combination M is also defined on the same factor space.
Thus M is representation independent.
All defining properties are satisfied, and therefore M is a compositional structural metric. ■

7.2 Structural Basis Property

The convex cone property allows the systematic construction of large families of compositional structural metrics.
In many cases structural metrics can be expressed as combinations of a small number of basic invariants.
In particular, many additive metrics on atomic forms can be expressed in terms of **length-like** and **depth-like** components.

Observation

For a broad class of additive and local structural metrics, the metric value can be decomposed into contributions from:

- the total number of computational vertices, and
- the structure of dependency chains.

Thus such metrics can often be expressed as combinations of invariants related to atomic length and atomic depth.

This observation should not be interpreted as a universal completeness statement.
Rather, it indicates that length and depth frequently serve as **structural coordinates** for families of compositional structural metrics.

Structural Interpretation

The convex cone structure implies that the class of compositional structural metrics is closed under scaling and combination.
Consequently, the set

$$\mathcal{M}_\Sigma$$

forms a convex cone in the vector space of real-valued functions on $\boldsymbol{A}_\Sigma$.
This means that compositional structural metrics form a structured family of invariants rather than isolated measurements.

7.3 Consequences

The convex cone structure of compositional structural metrics has several conceptual consequences.
First, it provides a systematic method for constructing new metrics by combining existing ones.
Second, it confirms that the space of admissible structural invariants is inherently **multi-dimensional**, reflecting the diverse structural aspects of dependency graphs.
Finally, this structure prepares the ground for the introduction of metric-like geometric structures on the space of atomic forms.
In the next section we show that every compositional structural metric naturally induces a pseudometric on the structural space of atomic forms.
This connection provides a geometric interpretation of structural complexity within the compositional framework.

8. Induced Pseudometric Structures

The compositional structural metrics introduced in the previous sections provide numerical measurements of structural complexity on the space of atomic forms.
In addition to their role as complexity measures, such metrics naturally induce geometric structures on the space of atomic forms.
In this section we show that every compositional structural metric defines a pseudometric on the factor space of atomic forms.
This construction provides a simple way to interpret structural differences between atomic forms in quantitative terms.
Importantly, the induced pseudometric reflects only the difference of structural measurements and does not depend on syntactic representations of computations.

8.1 Definition of the Induced Distance

Let

$$M: \boldsymbol{A}_\Sigma \to \mathbb{R}_{\geq 0}$$

be a compositional structural metric.
We define a function

$$d_M: \boldsymbol{A}_\Sigma \times \boldsymbol{A}_\Sigma \to \mathbb{R}_{\geq 0}$$

by

$$d_M(A_1, A_2) = |M(A_1) - M(A_2)|.$$

This quantity measures the absolute difference between the metric values of two atomic forms. Intuitively, $d_M(A_1, A_2)$ describes how far apart the two atomic forms are with respect to the structural measurement defined by M.

8.2 Pseudometric Property

We now verify that the function d_M satisfies the axioms of a pseudometric.

Theorem 8.1

Let M be a compositional structural metric.
Then the function d_M defined above is a pseudometric on the space $\boldsymbol{A}_\Sigma$.

Proof

To prove that d_M is a pseudometric, we verify the standard properties.

(1) Non-negativity

For all atomic forms A_1, A_2,

$$d_M(A_1, A_2) = |M(A_1) - M(A_2)| \geq 0.$$

(2) Symmetry

For all atomic forms A_1, A_2,

$$d_M(A_1, A_2) = |M(A_1) - M(A_2)| = |M(A_2) - M(A_1)| = d_M(A_2, A_1).$$

(3) Triangle inequality

Let A_1, A_2, A_3 be atomic forms.
Then

$$d_M(A_1, A_3) = |M(A_1) - M(A_3)|$$

By the triangle inequality for real numbers,

$$|M(A_1) - M(A_3)| \leq |M(A_1) - M(A_2)| + |M(A_2) - M(A_3)|$$

.

Therefore

$$d_M(A_1, A_3) \leq d_M(A_1, A_2) + d_M(A_2, A_3).$$

(4) Identity of indiscernibles (weak form)

If

$$A_1 = A_2$$

then clearly

$$d_M(A_1, A_2) = 0.$$

However, the converse need not hold: two non-isomorphic atomic forms may have identical values of the metric M.
Thus d_M is generally a **pseudometric** rather than a metric.
Since all pseudometric axioms are satisfied, the function d_M defines a pseudometric on $\boldsymbol{A}_\Sigma$. ■

8.3 Metric Quotients

Although d_M is in general only a pseudometric, it naturally induces a genuine metric on a quotient space.
Define an equivalence relation

$$A_1 \sim_M A_2 \Leftrightarrow M(A_1) = M(A_2).$$

In other words, two atomic forms are equivalent if they have identical values of the metric M.
Let

$$\boldsymbol{A}_\Sigma/\sim_M$$

denote the corresponding quotient space.

Proposition 8.1

The function dMd_MdM induces a metric on the quotient space $\boldsymbol{A}_\Sigma/\sim_M$.

Proof Sketch

If two equivalence classes are distinct, their metric values differ, and therefore the induced distance between them is strictly positive.
The pseudometric properties verified above therefore become the axioms of a genuine metric on the quotient space. ■

8.4 Behaviour under Composition

The pseudometric dMd_MdM is not required to be invariant under sequential composition.
In particular, in general we do not have

$$d_M(A_2 \circ A_1, B_2 \circ B_1) = d_M(A_1, B_1) + d_M(A_2, B_2).$$

Such an identity would impose very strong constraints on the metric and is not required in the definition of compositional structural metrics.
Nevertheless, the subadditivity property of CSMs ensures that the metric values grow in a controlled way under sequential composition.
Specifically, for composable atomic forms we have

$$M(A_2 \circ A_1) \leq M(A_1) + M(A_2).$$

Thus structural complexity measured by M cannot increase faster than linearly with respect to sequential composition.

8.5 Structural Interpretation

The pseudometric construction provides a simple geometric interpretation of compositional structural metrics.
Each metric M induces a one-dimensional geometric structure on the space of atomic forms, where the position of an atomic form is determined by the value of the invariant.
Different metrics therefore define different geometric views of the structural space.
Taken together, families of compositional structural metrics may be interpreted as defining coordinate systems for structural complexity.
This geometric viewpoint provides additional intuition for understanding the multi-dimensional nature of structural measurements.

8.6 Summary

Every compositional structural metric induces a pseudometric on the space of atomic forms.
These pseudometrics measure differences in structural complexity while remaining invariant under interface-preserving isomorphisms.
Although the induced pseudometrics are generally not invariant under composition, the subadditivity property of compositional structural metrics ensures controlled growth of structural measurements under sequential composition.
In the following section we interpret the fundamental structural metrics introduced earlier in terms of classical notions of work and dependency depth in parallel evaluation models.

9. Structural Interpretation and Relation to Parallel Evaluation

The compositional structural metrics introduced in the previous sections measure intrinsic properties of atomic dependency structures.
In particular, the invariants of atomic length and atomic depth capture two fundamental aspects of structural complexity.
These invariants admit a natural interpretation when atomic forms are viewed as representations of computational dependency graphs.
Although the present work does not introduce an execution model, the structural properties of atomic forms closely resemble the dependency structures studied in parallel computation.
This section discusses this structural relationship.

9.1 Atomic Length as Structural Work

Atomic length

$$L(A) = |V(A)|$$

counts the total number of atomic computational vertices in a dependency structure.
Since each vertex represents a single elementary operation from the signature Σ\SigmaΣ, atomic length measures the total number of atomic operations required by the computation.
From a structural perspective, atomic length corresponds to the amount of **total computational work** present in the dependency graph.
This interpretation parallels classical notions in computational models such as:

- circuit size in circuit complexity,
- program length in straight-line programs,
- work in parallel algorithm analysis.

However, in the present framework atomic length is interpreted purely as a **structural invariant of dependency graphs**, independent of any specific evaluation model.

9.2 Atomic Depth as Critical Dependency Path

Atomic depth

$$D(A)$$

measures the maximal length of a directed path in the dependency structure of an atomic form.
Such paths correspond to sequences of operations where each operation depends on the output of the previous one.
Consequently, atomic depth represents the longest chain of sequential dependencies in the computation.
In classical parallel computation models, the length of the longest dependency chain is often referred to as the **critical path length**.
This quantity determines the minimal number of sequential stages required to evaluate a computation when unlimited parallel resources are available.
Within the structural framework of atomic forms, atomic depth therefore measures the inherent **sequential dependency structure** of the computation.

9.3 Structural Parallelism

Combining atomic length and atomic depth yields a derived structural quantity often interpreted as a measure of parallelism.
Define

$$P(A) = \frac{L(A)}{D(A)}.$$

This quantity describes the average amount of work performed per dependency layer of the computation.
Intuitively, larger values of $P(A)$ correspond to structures where more operations can potentially be executed simultaneously once their dependencies are satisfied.
However, structural parallelism is not itself a compositional structural metric.
As discussed earlier, the ratio L/D generally fails to satisfy the monotonicity and subadditivity properties required of CSMs.
Therefore structural parallelism should be interpreted as a **derived invariant constructed from compositional structural metrics** rather than as a primary metric.

9.4 Structural Analogy with Parallel Evaluation

The quantities $L(A)$ and $D(A)$ correspond structurally to two classical measures studied in parallel algorithm theory:

- **work**, representing the total number of operations performed, and
- **critical path length**, representing the longest chain of dependencies.

In classical parallel evaluation theory, relations between work and critical path length play a central role in determining the efficiency of parallel algorithms.
For example, Brent's theorem establishes that computations with work W and depth T can often be scheduled on ppp processors in time approximately

$$O\left(\frac{W}{T} + T\right).$$

The structural invariants introduced in this paper resemble these quantities at the level of dependency graphs.
However, it is important to emphasize that the present work does **not introduce an execution model or complexity bounds**.
The interpretation here is purely structural: atomic forms encode dependency relationships, and the metrics L and D measure structural properties of these dependency graphs.

9.5 Structural Perspective

From the viewpoint of atomic forms, the quantities L and D describe two complementary dimensions of structural complexity:

- **global size** of the dependency structure, and
- **sequential depth** of the dependency chain.

Trade-offs between these quantities reflect different structural organizations of computations.
For instance:

- linear chains minimize structural width but maximize dependency depth,
- layered or tree-like structures reduce depth but often increase total size.

Such trade-offs arise naturally in many dependency structures and illustrate that structural complexity cannot be fully captured by a single invariant.

9.6 Summary

The compositional structural metrics introduced in this work provide a quantitative framework for studying atomic dependency structures.
Atomic length and atomic depth correspond structurally to the classical notions of total work and critical dependency path.
Although the present work does not address computational scheduling or performance, these analogies highlight the structural significance of the metrics introduced earlier.
In the final section we summarize the structural properties established in this work and state the main metric theorem describing the compositional structure of the space of atomic forms.

10. Conclusion

We introduced the class of compositional structural metrics on the space of atomic forms.
These metrics provide quantitative invariants compatible with the compositional algebra of atomic forms and the structural embedding preorder.
We showed that atomic length and atomic depth are fundamental examples of such metrics, that the class of CSMs forms a convex cone, and that every CSM induces a natural pseudometric on the space of atomic forms.
These results provide a quantitative framework for studying structural complexity of atomic dependency representations.

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